

17/04/08

Strong coupling for planar $\mathcal{N} = 4$ SYM theory: an all-order resultDavide Fioravanti ^a, Paolo Grinza ^b and Marco Rossi ^{c *}^a *Sezione INFN di Bologna, Dipartimento di Fisica, Università di Bologna,
Via Irnerio 46, Bologna, Italy*^b *Departamento de Fisica de Particulas, Instituto Galego de Fisica de Altas Enerxias
(IGFAE), E-15782, Universidade de Santiago de Compostela, Spain*^c *Dipartimento di Fisica, Università della Calabria, I-87036 Arcavacata di Rende,
Cosenza, Italy***Abstract**

We propose a scheme for determining a generalised scaling function, namely the Sudakov factor in a peculiar double scaling limit for high spin and large twist operators belonging to the $sl(2)$ sector of planar $\mathcal{N} = 4$ SYM. In particular, we perform explicitly the all-order computation at strong 't Hooft coupling regarding the first (contribution to the) generalised scaling function. Moreover, we compare our asymptotic results with the numerical solutions finding a very good agreement and evaluate numerically the non-asymptotic contributions. Eventually, we illustrate the agreement and prediction on the string side.

*E-mail: fioravanti@bo.infn.it, pgrinza@usc.es, rossi@cs.infn.it

1 On the set-up

The $sl(2)$ sector of $\mathcal{N} = 4$ SYM contains local composite operators of the form

$$\text{Tr}(\mathcal{D}^s \mathcal{Z}^L) + \dots, \quad (1.1)$$

where \mathcal{D} is the (symmetrised, traceless) covariant derivative acting in all possible ways on the L bosonic fields \mathcal{Z} . The spin of these operators is s and L is the so-called 'twist'. Moreover, this sector would be described – via the AdS/CFT correspondence [1] – by string states on the $\text{AdS}_5 \times \text{S}^5$ spacetime with AdS_5 and S^5 charges s and L , respectively. Moreover, as far as the one loop is concerned, the Bethe Ansatz problem is equivalent to that of twist operators in QCD [2, 3].¹

Proper superpositions of operators (1.1) have definite anomalous dimension Δ depending on L , s and the 't Hooft coupling $\lambda = 8\pi^2 g^2$:

$$\Delta = L + s + \gamma(g, s, L), \quad (1.2)$$

where $\gamma(g, s, L)$ is the anomalous part. A great boost in the evaluation of $\gamma(g, s, L)$ in another sector has come from the discovery of integrability for the purely bosonic $so(6)$ operators at one loop [4]. Later on, this fact has been extended to the whole theory and at all loops in the sense that, for instance, any operator of the form (1.1) is associated to one solution of some Bethe Ansatz-like equations and then any anomalous dimension is expressed in terms of one solution [5]. Nevertheless, along with this host of new results an important limitation emerged as a by-product of the *on-shell* (S-matrix) Bethe Ansatz: when the interaction range had been greater than the chain length, then unpredicted *wrapping effects* would have modified the same. More precisely, the anomalous dimension is *in general* correct up to the $L - 1$ loop in the convergent, perturbative expansion, i.e. up to the order g^{2L-2} . Which in particular implies, – fortunately for us, – that the *asymptotic* Bethe Ansatz gives the right result whenever the $L \rightarrow \infty$ limit is applied.

In this limit an important double scaling may be considered on both sides of the correspondence²:

$$s \rightarrow \infty, \quad L \rightarrow \infty, \quad j = \frac{L}{\ln s} = \text{fixed}. \quad (1.3)$$

In fact, the relevance of this logarithmic scaling for the anomalous dimension has been pointed out in [7] within the one-loop SYM theory and then in [6] and [8] within the string theory (strong coupling $g \gg 1$). Moreover, by describing the anomalous

¹A deep reason for that may be that one loop QCD still shows up conformal invariance, albeit integrability does not seem to hold in full generality (for instance, it apparently imposes aligned partonic helicities).

²Actually, in string theory (semi-classical calculations) the $\lambda \rightarrow +\infty$ limit needs consideration before any other (cf. for instance [6] and references therein), thus implying a different limit order with respect to our gauge theory approach (cf. below for more details).

dimension through a non-linear integral equation [9] (like in other integrable theories [10]), it has been recently conjectured the Sudakov scaling [8]

$$\gamma(g, s, L) = f(g, j) \ln s + O((\ln s)^{-\infty}), \quad (1.4)$$

thus indeed generalising to all loops the analogous result of [7].³ Actually, this statement was argued by computing iteratively the solution of some integral equations and then, thereof, *the generalised scaling function*, $f(g, j)$ at the first orders in j and g^2 : more precisely the first orders in g^2 have been computed for the first *generalised scaling functions* $f_n(g)$, forming

$$f(g, j) = \sum_{n=0}^{\infty} f_n(g) j^n. \quad (1.5)$$

As a by-product, the reasonable conjecture has been put forward that the two-variable function $f(g, j)$ should be analytic (in g for fixed j and in j for fixed g). In [11] similar results have been derived for what concerns the contribution beyond the leading scaling function $f(g) = f_0(g)$, but with a modification (see also [12] for the general idea) which has allowed us to neglect the non-linearity for finite L and to end-up with one linear integral equation. The latter does not differ from the BES one (which cover the case $j = 0$, cf. the last of [5]), but for the inhomogeneous term, which is an integral on the one loop root density. In this respect, we have thought interesting the analysis of the next-to-leading-order (nlo) term – still coming, for finite L , from an asymptotic Bethe Ansatz –, as the leading order $f(g)$ has been conjectured to be independent of L or *universal* (penultimate reference of [5], after the one loop proof by [7]).

For instance, the linearity in L of this nlo term and other features have furnished us the *inspiration* for the present calculations. In this respect, we deem useful to briefly introduce in the next Section a suitable modification of the aforementioned method such that it applies to the regime (1.3) (still for any g and j). In principle, we might in this way determine the generalised scaling function $f(g, j)$, and also its *constituents* $f_n(g)$. As a consequence of the full range of g we can work out not only the weak coupling expansion, but also the exact strong coupling evaluations (here only for the simplest case concerning $f_1(g)$), and to perform an *all g* numerical work. Eventually, we shall highlight the concise style of the present publication, in that a next one with much more details shall be coming out rather soon.

2 Computing the generalised scaling function

Because of integrability in $\mathcal{N} = 4$ SYM, we have been finding useful [13] to rewrite the Bethe equations as non-linear integral equations, whose integration range is, even for massive excited states (holes and complex roots), $(-\infty, +\infty)$ [10]. More recently

³ $O((\ln s)^{-\infty})$ means a remainder which goes faster than any power of $\ln s$: $\lim_{s \rightarrow \infty} (\ln s)^k O((\ln s)^{-\infty}) = 0, \forall k > 0$.

[11, 12], we have developed a new technique in order to cope with an integration interval that may be kept finite, having in mind the peculiar case of the $sl(2)$ sector with two holes external to all the roots [7, 5, 9]. With reference in the entire Section to [11], the non-linear integral equation for the $sl(2)$ sector involves two functions $F(u)$ and $G(u, v)$ satisfying linear equations. Splitting $F(u)$ into its one-loop and higher loop contributions, $F_0(u)$ and $F^H(u)$ respectively, the latter has been shown to satisfy (cf. (4.10) and (4.11) of [11]):

$$\begin{aligned}
\sigma_H(u) = & -iL \frac{d}{du} \ln \left(\frac{1 + \frac{g^2}{2x^-(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right) + \\
& - \frac{i}{\pi} \int_{-b_0}^{b_0} dv \frac{d}{du} \left[\ln \left(\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right] [\sigma_0(v) + 2\pi\delta_h(v)] + \\
& + \int_{-\infty}^{+\infty} \frac{dv}{\pi} \frac{1}{1 + (u - v)^2} [\sigma_H(v) + 2\pi\eta_h(v)] - \\
& - \frac{i}{\pi} \int_{-\infty}^{+\infty} dv \frac{d}{du} \left[\ln \left(\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right] \sigma_H(v) + O((\ln s)^{-\infty}),
\end{aligned} \tag{2.1}$$

which holds when $s \rightarrow \infty$ once introduced in equation (4.10) of [11] the Bethe roots densities ⁴ $\sigma_H(u) = \frac{d}{du} F^H(u)$, g -dependent, and $\sigma_0(u) = \frac{d}{du} F_0(u)$, g -independent ⁵. Here the hole contributions given by $\delta_h(v) = (L - 2)\delta(v) + O(L^2/\ln s)$ and $\eta_h(v) = O(L^2/\ln s)$ depend on the logarithmic scaling to zero of the internal holes. Moreover, the solution of (2.1) yields the anomalous dimension in the large spin limit while neglecting the non-linear convolution term ⁶:

$$\begin{aligned}
\gamma(g, s, L) = & -g^2 \int_{-b_0}^{b_0} \frac{dv}{2\pi} \left[\frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] [\sigma_0(v) + 2\pi\delta_h(v)] - \\
& - g^2 \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] \sigma_H(v) + O((\ln s)^{-\infty}) =
\end{aligned} \tag{2.2}$$

$$= \frac{1}{\pi} \hat{\sigma}_H(k = 0) + O((\ln s)^{-\infty}), \tag{2.3}$$

where the last equality is a generalisation of the analogue in [14].

As we are just neglecting terms smaller than any inverse logarithm, $O((\ln s)^{-\infty})$, we can efficiently consider the double scaling limit (1.3). Looking at (2.1), the r.h.s. is made up of seven terms, among which the first is explicit and the last uninfluential. On the contrary, the third and the fifth integral reflect two different kinds of important hole contributions. Nevertheless, thanks to the quoted scaling $\delta_h(v) = (L - 2)\delta(v) + O(L^2/\ln s)$ and $\eta_h(v) = O(L^2/\ln s)$, these integrals enjoy an expansion about $j = 0$

⁴The actual positive densities would also have a factor $-1/(2\pi)$ in front [11].

⁵The dependence $b_0(s) = s/2 + O(s^0)$ comes out from the normalisation of the density $\sigma_0(u)$ [11].

⁶We also have some numerical evidence supporting this fact.

(similar to the following one) contributing only with the Dirac delta function to $f_1(g)$, and the remainder (in particular the all $\eta_h(v)$) to the higher $f_n(g)$, $n = 2, 3, \dots$. And last but not least, the second term determines the inhomogeneous (or forcing) term as a series about $j = 0$ via the density $\sigma_0(u)$, by virtue of the *logarithmic expansion* [7, 9] of the one-loop theory [11]:

$$-\frac{i}{\pi} \int_{-b_0}^{b_0} dv \frac{d}{du} \left[\ln \left(\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right] \sigma_0(v) = \left[\sum_{n=0}^{\infty} \phi_n(u) j^n \right] \ln s + O((\ln s)^{-\infty}). \quad (2.4)$$

In fact, this series shall follow from the solution (in the scaling (1.3)) of the linear integral equation for the one loop density $\sigma_0(u)$ (equation (4.9) of [11]), which determines all the functions $\phi_n(u)$. As a consequence, the solution of (2.1) inherits the same form of the forcing term

$$\sigma_H(u) = \left[\sum_{n=0}^{\infty} \sigma_H^{(n)}(u) j^n \right] \ln s + O((\ln s)^{-\infty}). \quad (2.5)$$

Eventually, all the generalised scaling functions $f(g, j)$ and $f_n(g)$ may be computed, in principle, via the equality (2.3). From now on, we will restrict ourselves to the first order beyond the $j = 0$ theory [5], i.e. $\sigma_H^{(1)}(u)$ and $f_1(g)$.

3 The first generalised scaling function

In [11] we have found a way to compute the Stieltjes integrals with measure induced by the one loop-density $\sigma_0(v)$ on the support $(-b_0, b_0)$ via an *effective* density $\sigma_0^{(b_0)}(v)$ on support $(-\infty, +\infty)$

$$\int_{-b_0}^{b_0} dv \sigma_0(v) q(v) = \int_{-\infty}^{\infty} dv \sigma_0^{(b_0)}(v) q(v) + O(1/\ln s), \quad (3.1)$$

where the Fourier transform of the function $\sigma_0^{(b_0)}(v)$ reads explicitly

$$\hat{\sigma}_0^{(b_0)}(k) = -4\pi \frac{\frac{L}{2} - e^{-\frac{|k|}{2}} \cos \frac{ks}{\sqrt{2}}}{2 \sinh \frac{|k|}{2}} + 2\pi(L-2) \frac{e^{-\frac{|k|}{2}}}{2 \sinh \frac{|k|}{2}} - 4\pi \delta(k) \ln 2, \quad (3.2)$$

and $q(v)$ is a suitable test-function. Therefore, we can compute exactly the functions $\phi_0(u)$ and $\phi_1(u)$ and thus write down the linear integral equations for $\sigma_H^{(0)}(u)$ and $\sigma_H^{(1)}(u)$ respectively. Of course, the equation for $\sigma_H^{(0)}(u)$ is the BES equation [5], whilst that about $\sigma_H^{(1)}(u)$ is a novelty. We may find still convenient to manipulate the Fourier transforms defining the even function

$$s(k) = \frac{2 \sinh \frac{|k|}{2}}{2\pi |k|} \hat{\sigma}_H^{(1)}(k), \quad (3.3)$$

which has the important property (induced by (2.3))

$$s(0) = \frac{1}{2}f_1(g). \quad (3.4)$$

If we introduce for convenience the functions

$$a_r(g) = \int_{-\infty}^{+\infty} \frac{dh}{h} J_r(\sqrt{2}gh) \frac{1}{1 + e^{\frac{|h|}{2}}}, \quad \bar{a}_r(g) = \int_{-\infty}^{+\infty} \frac{dh}{|h|} J_r(\sqrt{2}gh) \frac{1}{1 + e^{\frac{|h|}{2}}}, \quad (3.5)$$

and re-cast the integration inside the domain $k \geq 0$, the integral equation for $s(k)$ takes the form

$$\begin{aligned} s(k) &= \frac{1 - J_0(\sqrt{2}gk)}{k} - \sum_{p=1}^{\infty} 2p \bar{a}_{2p}(g) \frac{J_{2p}(\sqrt{2}gk)}{k} - \sum_{p=1}^{\infty} (2p-1) a_{2p-1}(g) \frac{J_{2p-1}(\sqrt{2}gk)}{k} \\ &+ 2 \sum_{p=1}^{\infty} \sum_{\nu=0}^{\infty} \left[(-1)^{1+\nu} c_{2p+1, 2p+2\nu+2}(g) a_{2p+2\nu+1}(g) \frac{J_{2p}(\sqrt{2}gk)}{k} + \right. \\ &+ \left. (-1)^{1+\nu} c_{2p, 2p+2\nu+1}(g) a_{2p-1}(g) \frac{J_{2p+2\nu}(\sqrt{2}gk)}{k} \right] - \\ &- 2 \sum_{p=1}^{\infty} 2p \frac{J_{2p}(\sqrt{2}gk)}{k} \int_0^{\infty} dh \frac{J_{2p}(\sqrt{2}gh)}{e^h - 1} s(h) - \\ &- 2 \sum_{p=1}^{\infty} (2p-1) \frac{J_{2p-1}(\sqrt{2}gk)}{k} \int_0^{\infty} dh \frac{J_{2p-1}(\sqrt{2}gh)}{e^h - 1} s(h) + \\ &+ 4 \sum_{p=1}^{\infty} \sum_{\nu=0}^{\infty} \left[(-1)^{1+\nu} c_{2p+1, 2p+2\nu+2}(g) \frac{J_{2p}(\sqrt{2}gk)}{k} \int_0^{\infty} dh \frac{J_{2p+2\nu+1}(\sqrt{2}gh)}{e^h - 1} s(h) \right. \\ &+ \left. (-1)^{1+\nu} c_{2p, 2p+2\nu+1}(g) \frac{J_{2p+2\nu}(\sqrt{2}gk)}{k} \int_0^{\infty} dh \frac{J_{2p-1}(\sqrt{2}gh)}{e^h - 1} s(h) \right]. \end{aligned} \quad (3.6)$$

As anticipated, this equation shares the same kernel with BES equation [5], but different forcing term. Its solution may be expanded in a series involving Bessel functions:

$$\begin{aligned} s(k) &= \frac{1 - J_0(\sqrt{2}gk)}{k} + \sum_{p=1}^{\infty} s_{2p}(g) \frac{J_{2p}(\sqrt{2}gk)}{k} + \sum_{p=1}^{\infty} s_{2p-1}(g) \frac{J_{2p-1}(\sqrt{2}gk)}{k} = \\ &= \sum_{p=1}^{\infty} (2 + s_{2p}(g)) \frac{J_{2p}(\sqrt{2}gk)}{k} + \sum_{p=1}^{\infty} s_{2p-1}(g) \frac{J_{2p-1}(\sqrt{2}gk)}{k}. \end{aligned} \quad (3.7)$$

Now we want to derive linear equations for the coefficients s_n . Upon expressing the scattering factor coefficients via

$$Z_{n,m}(g) = \int_0^{\infty} \frac{dh}{h} \frac{J_n(\sqrt{2}gh) J_m(\sqrt{2}gh)}{e^h - 1}, \quad (3.8)$$

in the form

$$c_{r,s}(g) = 2 \cos \left[\frac{\pi}{2}(s - r - 1) \right] (r-1)(s-1) Z_{r-1, s-1}(g), \quad (3.9)$$

equation (3.6) decomposes into the infinite dimensional linear system

$$S_{2p}(g) = 2 + 2p \left(-\bar{a}_{2p}(g) - 2 \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}(g) + 2 \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) s_{2m-1}(g) \right) \quad (3.10)$$

$$\frac{s_{2p-1}(g)}{2p-1} = -a_{2p-1}(g) - 2 \sum_{m=1}^{\infty} Z_{2p-1,2m}(g) S_{2m}(g) - 2 \sum_{m=1}^{\infty} Z_{2p-1,2m-1}(g) s_{2m-1}(g),$$

where we have re-defined for brevity

$$S_{2m}(g) = 2 + s_{2m}(g). \quad (3.11)$$

4 Exact asymptotic expansion

At this point, we start looking for the solution of equations (3.10) for $g \rightarrow +\infty$ in the form of an asymptotic series, i.e.

$$s_{2m}(g) \doteq \sum_{n=0}^{\infty} \frac{s_{2m}^{(n)}}{g^n}, \quad s_{2m-1}(g) \doteq \sum_{n=0}^{\infty} \frac{s_{2m-1}^{(n)}}{g^n}. \quad (4.1)$$

Albeit we well know that the unique knowledge of the asymptotic expansion will never identify the function itself (contrary to the convergent weak-coupling series [9, 11]), we will see in the following many favourable issues, besides the contact with semi-classical string theory results. From the explicit solution of (3.10) up to the order $1/g^2$, we have guessed the form at all orders:

$$s_{2m}^{(2n)} = 2m \frac{\Gamma(m+n)}{\Gamma(m-n+1)} (-1)^{1+n} b_{2n}, \quad s_{2m-1}^{(2n)} = 0; \quad n \geq 0, \quad m \geq 1, \quad (4.2)$$

$$s_{2m}^{(2n-1)} = 0, \quad s_{2m-1}^{(2n-1)} = (2m-1) \frac{\Gamma(m+n-1)}{\Gamma(m-n+1)} (-1)^n b_{2n-1}; \quad n \geq 1, \quad m \geq 1,$$

with, implicitly, these expressions different from zero only if $n \leq m$. Moreover, the coefficients b_n satisfy a recursive relation whose solution is more easily written in terms of their generating function

$$b(t) = \sum_{n=0}^{\infty} b_n t^n, \quad (4.3)$$

which is then worth

$$b(t) = \frac{1}{\cos \frac{t}{\sqrt{2}} - \sin \frac{t}{\sqrt{2}}}. \quad (4.4)$$

Consequently, we may also derive the explicit values

$$b_{2n} = 2^{-n}(-1)^n \sum_{k=0}^n \frac{E_{2k} 2^{2k}}{(2k)!(2n-2k)!}, \quad (4.5)$$

$$b_{2n-1} = 2^{-n+\frac{1}{2}}(-1)^{n-1} \sum_{k=0}^{n-1} \frac{E_{2k} 2^{2k}}{(2k)!(2n-2k-1)!},$$

where E_{2k} are Euler's numbers. To summarise equations (4.1, 4.2, 4.5), we obtain the following strong coupling solution to the linear system (3.10):

$$s_{2m-1}(g) \doteq -(2m-1) \sum_{n=1}^m \frac{\Gamma(m+n-1)}{\Gamma(m-n+1)} \frac{2^{\frac{1}{2}-n}}{g^{2n-1}} \sum_{k=0}^{n-1} \frac{E_{2k} 2^{2k}}{(2k)!(2n-2k-1)!}, \quad (4.6)$$

$$s_{2m}(g) \doteq -2m \sum_{n=0}^m \frac{\Gamma(m+n)}{\Gamma(m-n+1)} \frac{2^{-n}}{g^{2n}} \sum_{k=0}^n \frac{E_{2k} 2^{2k}}{(2k)!(2n-2k)!}.$$

As these series truncate, they are *asymptotic* in a very peculiar sense, namely they are correct up to non-expandable corrections which still go to zero (cf. below). From definition (3.7) and property (3.4) we can read off

$$s(0) = \frac{g}{\sqrt{2}} s_1(g) = \frac{1}{2} f_1(g), \quad (4.7)$$

which entails the exact asymptotic expression $f_1(g) \doteq -1$.

It is very interesting to provide a possible summation for the expansions (4.6) (though finite!) by using an integral representation for the ratio of the two gamma functions and then by exchanging the sum over n with the integral, and finally summing up: this procedure provides these *new* functions of g

$$\tilde{s}_{2m-1}(g) = -2(2m-1) \int_0^\infty \frac{dt}{t} \frac{\sinh t}{\cosh 2t} J_{2m-1}(2\sqrt{2}gt) \quad (4.8)$$

$$\tilde{s}_{2m}(g) = -4m \int_0^\infty \frac{dt}{t} \frac{\cosh t}{\cosh 2t} J_{2m}(2\sqrt{2}gt).$$

Instead of the first generalised scaling function we obtain

$$\tilde{f}_1(g) = -2\sqrt{2}g \int_0^\infty \frac{dt}{t} \frac{\sinh t}{\cosh 2t} J_1(2\sqrt{2}gt). \quad (4.9)$$

From this integral representation we see that the asymptotic behaviour becomes corrected by exponentially small terms, behaving as $g^{1/2} \exp(-\pi g/\sqrt{2})$. We will see in next section that this is not quite the right behaviour because of the wrong power $g^{1/2}$. This is obviously connected to the wrong (convergent) weak coupling expansion of (4.9).

5 Numerical solution: the non-asymptotic terms

Remarkably, the linear system (3.10) shows up the same matrix as in [15] for the BES equation, but a more involved forcing term. Explicitly, it may assume the form

$$s_p(g) = b_p(g) - \sum_{m=1}^{\infty} (K_{pm}^{(m)}(g) + 2K_{pm}^{(c)}(g))s_m(g), \quad (5.10)$$

where

$$\begin{aligned} K_{pm}^{(m)}(g) &= 2(NZ)_{pm}, & K_{pm}^{(c)}(g) &= 4(PNZQNZ)_{pm} \\ b(g) &= -(N + 4PNZQN)a^T - N\bar{a}^T - 4(NZP + 4PNZQNZP)t^T, \end{aligned} \quad (5.11)$$

with

$$\begin{aligned} N &= \text{diag}(1, 2, 3, \dots), & P &= \text{diag}(0, 1, 0, 1, \dots), & Q &= \text{diag}(1, 0, 1, 0, \dots), \\ t &= (1, 0, 1, 0, \dots), & a &= (a_1, 0, a_3, 0, \dots), & \bar{a} &= (0, \bar{a}_2, 0, \bar{a}_4, \dots). \end{aligned} \quad (5.12)$$

At least formally, the solution may be written as

$$s(g) = (\mathcal{I} + K^{(m)} + 2K^{(c)})^{-1} b, \quad (5.13)$$

where \mathcal{I} is the identity matrix. This similarity with [15] is crucial for what concerns the numerical treatment, in that everything works well (as in [15]), because (the matrix elements are the same and) the forcing elements have a similar behaviour for large g .

In the following we will be interested in a numerical analysis of (5.13), emphasising the deviation of the strong coupling behaviour of the first few $s_m(g)$ from the exact expansion of the previous Section. In this respect, a sufficient accuracy is achieved upon truncating at $m = 30$: this restriction still yields reliable results up to about $g \simeq 20$.

5.1 The strong coupling corrections and the string side

Since the asymptotic expansion for any $s_p(g)$ truncates significantly at the (inverse) power $1/g^p$, the additional (non-perturbative) exponential corrections are simply given by subtracting this to the numerical solution of (5.13)⁷. At any rate, our main concern here is $s_1(g)$ by virtue of its relation (4.7) to the generalised scaling function $f_1(g)$, whose asymptotic expansion $f(g) \doteq -1$ is in perfect agreement with the string energy density $\Delta - s = \gamma + L$ (cf. (1.2)), as perturbatively expanded in [6, 8], because at present it does not seem to have the term L . Moreover, beyond this asymptotics there are additional, non-perturbative, exponential corrections, whose leading order (see below (5.14)) is reported in figure 1, where the large g , asymptotic behaviour is already reached for $g \sim 3.5$.

⁷At the moment, we observe the same exponential behaviour, with different constant factors (cf. below), for the first few $s_n(g)$, $n = 1, \dots, 5$.

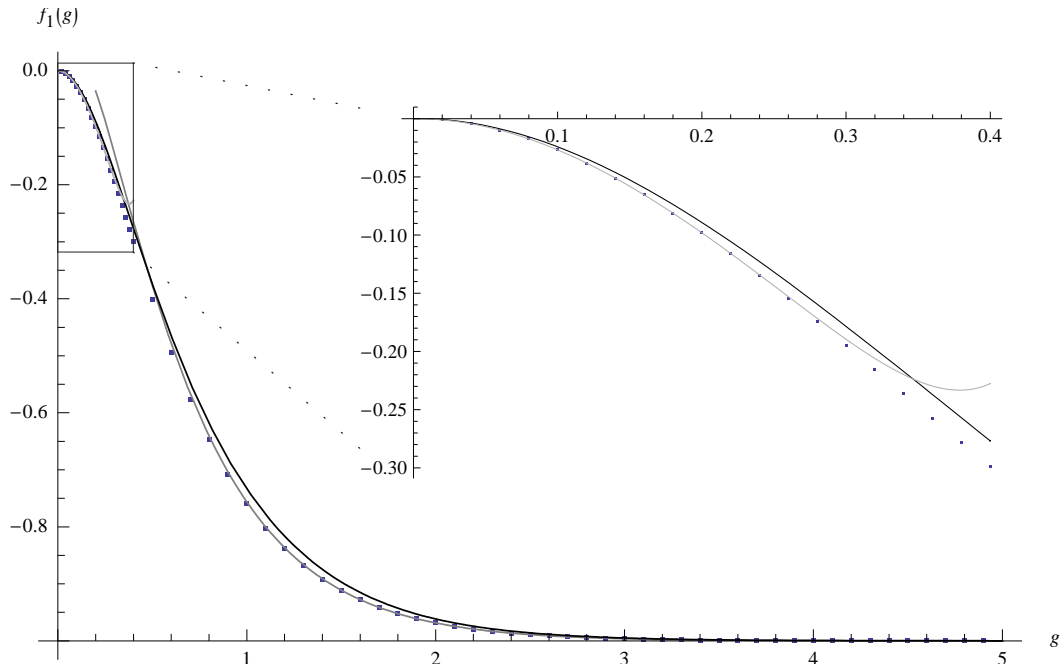


Figure 1: Comparison between numerical solution (dots), fit as in formula (5.14) (dark grey line) and integral representation (4.9) (black line). Detail in the small window: comparison between numerical solution (dots), weak coupling expansion of [9, 11] (light grey line) and integral representation (4.9) (black line).

We can then proceed to a more quantitative analysis. At first, we perform a fit of the numerical evaluation of (5.13), in order to check whether it is compatible with the proposed exact asymptotic expansion (4.6). This analysis – performed at the moment on the first few $s_n(g)$, $n = 1, \dots, 5$ – rules out the possibility that other terms with the form $1/g^k$ might appear. Then, we find reasonable to conjecture a non-perturbative, exponential term coming into play. In fact, a simple dimensional consideration has led the authors of [8] to claim that, regarding $f_1(g)$, the additional string energy density⁸ is simply proportional to $(j$ and) the $O(6)$ sigma model mass gap, m , or in formulæ $(f_1(g) + 1)j = cm(g)j$ as long as $j \ll m$ (still $g \rightarrow +\infty$).⁹ As a consequence, we use the functional form implied by the mass gap formula ([8] and reference [13] therein), i.e.

$$f_1(g) = -1 + \kappa g^{1/4} e^{-\frac{\pi}{\sqrt{2}}g}, \quad g \rightarrow \infty, \quad (5.14)$$

with only κ as a free parameter to fit, and we find a perfect agreement with the data

⁸This has to be summed to $f(g) \ln s$ and is due to the $SO(6)$ charge j .

⁹We may expect that $m(g)$ in this formula differs from the mass gap as soon as $j \ll g$ is no longer valid.

in the region $g \in [3, 20]$, resulting in the best fit estimate $\kappa = 2.257 \pm 0.009$. On the other hand, the $O(6)$ small coupling calculations of [8] entail the exact value $\kappa = 2^{5/8} \pi^{1/4} c / \Gamma(5/4) = (2.265218666...) c$, which compels us to the natural prediction that $c = 1$.¹⁰

We can now comment a little about the integral summation (4.9) to $\tilde{f}_1(g)$. Despite the fact that it is able to qualitatively capture the main features of the numerical solution for a wide range of g , it still fails to reproduce both the weak-coupling expansion, and the power factor $g^{1/4}$ of the non-perturbative term. We have a clear picture of this situation in fig. 1. Finally, we shall remark the perfect agreement between the numerical solution and the first terms of the weak-coupling expansion computed in [9, 11].

6 Some final comments

Unlike all the genuine (i.e. infinite) asymptotic series involved in the derivation of the leading term $f(g)$ (see, for instance, [16]), which amounts to twice the cusp anomalous dimension [17], all the analogous series for $f_1(g)$ truncate, thus giving rise to polynomials in $1/g$. Moreover, the latter become non-trivially corrected by exponentially small contributions. At this stage, a deeper comparison with string results ([6] and references therein) is really compelling, but still demanding because of the summation of all the logarithmic contributions in the string loop expansion [6]. Yet, this is the direct consequence of an ordering of different limits and the present results shed some light on what happens to the higher f_n , $n = 2, 3, \dots$.

Acknowledgments D.F. thanks D. Bombardelli, F. Buccheri, E. Caliceti, F. Canata, F. Ravanini, R. Soldati for discussions, the INFN grant "Iniziativa specifica PI14" and the international agreement INFN-MEC-2008 for travel financial support. P.G. work is partially supported by MEC-FEDER (grant FPA2005-00188), by the Spanish Consolider-Ingenio 2010 Programme CPAN (CSD2007-00042) and by Xunta de Galicia (Consellería de Educación and grant PGIDIT06PXIB296182PR). M.R. thanks INFN-Bologna for kind hospitality.

References

- [1] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 and hep-th/9711200; • S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys.Lett. **B428** (1998) 105 and hep-th/9802109; • E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253 and hep-th/9802150;

¹⁰In this respect there is also an ongoing analytic evaluation.

- [2] L.N. Lipatov, *Evolution equations in QCD*, in “Perspectives in Hadron Physics”, Proceedings of the Conference, ICTP, Trieste, Italy, May 1997, World Scientific (Singapore, 1998);
- [3] V.M. Braun, S.E. Derkachov, A.N. Manashov, *Integrability of three particle evolution equations in QCD*, Phys. Rev. Lett. **81** (1998) 2020 and hep-ph/9805225;
 - V.M. Braun, S.E. Derkachov, G.P. Korchemsky, A.N. Manashov, *Baryon distribution amplitudes in QCD*, Nucl. Phys. **B553** (1999) 355 and hep-ph/9902375;
 - A.V. Belitsky, *Fine structure of spectrum of twist-three operators in QCD*, Phys. Lett. **B453** (1999) 59 and hep-ph/9902361;
 - A.V. Belitsky, *Renormalization of twist-three operators and integrable lattice models*, Nucl. Phys. **B574** (2000) 407 and hep-ph/9907420;
 - A.V. Belitsky, A.S. Gorsky, G.P. Korchemsky, *Gauge / string duality for QCD conformal operators*, Nucl. Phys. **B667** (2003) 3 and hep-th/0304028;
- [4] J.A. Minahan, K. Zarembo, *The Bethe Ansatz for $\mathcal{N} = 4$ Super Yang-Mills*, JHEP**03** (2003) 013 and hep-th/0212208;
- [5] N. Beisert, C. Kristjansen, M. Staudacher, *The dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory*, Nucl. Phys. **B664** (2003) 131 and hep-th/0303060;
 - N. Beisert, M. Staudacher, *The $\mathcal{N} = 4$ SYM integrable super spin chain*, Nucl. Phys. **B670** (2003) 439 and hep-th/0307042;
 - N. Beisert, M. Staudacher, *Long-range PSU(2, 2|4) Bethe Ansatz for gauge theory and strings*, Nucl. Phys. **B727** (2005) 1 and hep-th/0504190;
 - B. Eden, M. Staudacher, *Integrability and transcendentality*, J.Stat.Mech. **11** (2006) P014 and hep-th/0603157;
 - N. Beisert, B. Eden, M. Staudacher, *Transcendentality and crossing*, J.Stat.Mech. **07** (2007) P01021 and hep-th/0610251;
- [6] S. Frolov, A. Tirziu, A.A. Tseytlin, *Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT*, Nucl. Phys. **B766** (2007) 232 and hep-th/0611269;
 - P.Y. Casteill, C. Kristjansen, *The Strong Coupling Limit of the Scaling Function from the Quantum String Bethe Ansatz*, Nucl. Phys. **B785** (2007) 1 and arXiv:0705.0890 [hep-th];
 - R. Roiban, A.A. Tseytlin, *Spinning superstrings at two loops: strong-coupling corrections to dimensions of large-twist SYM operators*, Phys.Rev. **D77** (2008) 066006 and arXiv:0712.2479 [hep-th];
- [7] A.V. Belitsky, A.S. Gorsky, G.P. Korchemsky, *Logarithmic scaling in gauge/string correspondence*, Nucl. Phys. **B748** (2006) 24 and hep-th/0601112;
- [8] L.F. Alday, J.M. Maldacena, *Comments on operators with large spin*, JHEP**11** (2007) 019 and arXiv:0708.0672 [hep-th];
- [9] L. Freyhult, A. Rej, M. Staudacher, *A Generalized Scaling Function for AdS/CFT*, arXiv:0712.2743;

- [10] D. Fioravanti, A. Mariottini, E. Quattrini, F. Ravanini, *Excited state Destri-de Vega equation for sine-Gordon and restricted sine-Gordon models*, Phys. Lett. **B390** (1997) 243 and hep-th/9608091; • C. Destri, H.J. de Vega, *Non linear integral equation and excited states scaling functions in the sine-Gordon model*, Nucl. Phys, **B504** (1997) 621 and hep-th/9701107;
- [11] D. Bombardelli, D. Fioravanti, M. Rossi, *Large spin corrections in $\mathcal{N} = 4$ SYM $sl(2)$: still a linear integral equation*, arXiv:0802.0027;
- [12] D. Bombardelli, D. Fioravanti, M. Rossi, *Non-linear integral equations in $\mathcal{N} = 4$ SYM*, Proceedings of International Workshop on Recent Advances in Quantum Integrable Systems (RAQIS 07), Annecy-le-Vieux, France, 11-14 Sep 2007 and arXiv:0711.2934 [hep-th];
- [13] G. Feverati, D. Fioravanti, P. Grinza, M. Rossi, *On the finite size corrections of anti-ferromagnetic anomalous dimensions in $\mathcal{N} = 4$ SYM*, JHEP**05** (2006) 068 and hep-th/0602189; • G. Feverati, D. Fioravanti, P. Grinza, M. Rossi, *Hubbard's Adventures in $\mathcal{N} = 4$ SYM-land? Some non-perturbative considerations on finite length operators*, J.Stat.Mech. **02** (2007) P001 and hep-th/0611186; • D. Fioravanti, M. Rossi, *On the commuting charges for the highest dimension $SU(2)$ operators in planar $\mathcal{N} = 4$ SYM*, JHEP**08** (2007) 089 and arXiv:0706.3936 [hep-th];
- [14] A.V. Kotikov, L.N. Lipatov, *On the highest transcendentality in $N=4$ SUSY*, Nucl. Phys, **B769** (2007) 217 and hep-th/0611204;
- [15] M. K. Benna, S. Benvenuti, I. R. Klebanov, A. Scardicchio, *A Test of the AdS/CFT Correspondence Using High-Spin Operators*, Phys. Rev. Lett. **98** (2007) 131603 and hep-th/0611135; • L. F. Alday, G. Arutyunov, M. K. Benna, B. Eden, I. R. Klebanov, *On the Strong Coupling Scaling Dimension of High Spin Operators*, JHEP**04** (2007) 082 and hep-th/0702028;
- [16] B. Basso, G.P. Korchemsky, J. Kotanski, *Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling*, Phys. Rev. Lett. **100** (2008) 091601 and arXiv:0708.3933 [hep-th]; • I. Kostov, D. Serban, D. Volin, *Functional BES equation*, arXiv:0801.2542;
- [17] G.P. Korchemsky, G. Marchesini, *Partonic distributions for large x and renormalization of Wilson loops*, Nucl. Phys. **B406** (1993) 225 and hep-ph/9210281; • G.P. Korchemsky, *Asymptotics of the Altarelli-Parisi-Lipatov Evolution Kernels of Parton Distributions*, Mod. Phys. Lett. **A4** (1989) 1257.